Solving Systems of 3 Equations using row-echelon form

Begin with a System of 3 Equations

$$\begin{aligned} x - 3y + 2z &= 9 \\ 2x + 5y - z &= -10 \\ -3x + y - 4z &= -5 \end{aligned}$$

Put coefficients in an augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 9 \\ 2 & 5 & -1 & -10 \\ -3 & 1 & -4 & -5 \end{array}\right]$$

Use row operations to get into row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 9 \\ 2 & 5 & -1 & -10 \\ -3 & 1 & -4 & -5 \end{array}\right]$$

Turn these into zeros

Turn these into ones

Row operations involve adding, subtracting, multiplying or dividing to change the entries in the row.

Begin by taking 2 times Row 1 and subtracting Row 2, creating a new Row 2

$2R_1 - R_2 \rightarrow R_2$

$2R_1$	2	-6	4	18
$-R_2$	-2	-5	+1	+10
New R_2	0	-11	5	28

Use Row 1 again. Add 3 times the entries in Row 1 to Row 3, creating a new Row 3

$3R_1 + R_3 \rightarrow R_3$

$3R_1$	3	-9	6	27
$+R_3$	-3	+1	-4	-5
New R_3	0	-8	2	22

Take the new R_2 and new R_3 and write out the new matrix.

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 9 \\ 0 & -11 & 5 & 28 \\ 0 & -8 & 2 & 22 \end{array}\right]$$

Divide R_2 to get a '1' in the second column of that row.

$R_2 \div -11 \rightarrow R_2$

$R_2/-11$	$\frac{0}{-11}$	$\frac{-11}{-11}$	$\frac{5}{-11}$	$\frac{28}{-11}$
New R_2	0	1	-5/11	-28/11

Write out the new matrix, noting R_3 needs changes.

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 9 \\ 0 & 1 & -\frac{5}{11} & -\frac{28}{11} \\ 0 & -8 & 2 & 22 \end{array}\right]$$

Add 8 times the entries in Row 2 to Row 3, creating a new Row 3.

$8R_2 + R_3 \rightarrow R_3$

$8R_2$	0	8	-40/11	-224/11
$+R_3$	0	-8	2	22
New R_3	0	0	-18/11	18/11

Write out the matrix with the new Row 3.

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 9 \\ 0 & 1 & -\frac{5}{11} & -\frac{28}{11} \\ 0 & 0 & -\frac{18}{11} & \frac{18}{11} \end{array}\right]$$

Multiply R_3 to get a '1' in the third column of that row.

$R_3 \times -11/18 \rightarrow R_3$

$R_3 \times -\frac{11}{18}$	0	0	$-\frac{18}{11}\left(\frac{-11}{18}\right)$	$\frac{18}{11}\left(\frac{-11}{18}\right)$
New R_3	0	0	1	-1

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Solving Systems of 3 Equations using row-echelon form

Now the matrix is in row-echelon form, with zeros and ones where they should be.

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 9 \\ 0 & 1 & -\frac{5}{11} & -\frac{28}{11} \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Convert the matrix back into equations with variables.

$$x - 3y + 2z = 9$$
$$0x + y - \frac{5}{11}z = -\frac{28}{11}$$
$$0x + 0y + z = -1$$

Use the third equation result to substitute in the second equation and solve for y. Use the y and z values and substitute in the first equation to solve for x.

$$z = -1$$

(Second Equation)

$$y - \frac{5}{11}z = -\frac{28}{11}$$
$$y - \frac{5}{11}(-1) = -\frac{28}{11}$$
$$y + \frac{5}{11} = -\frac{28}{11}$$
$$y = -\frac{33}{11} = -3$$

$$z = -1 \text{ and } y = -3$$

(First Equation)

$$x - 3y + 2z = 9$$
$$x - 3(-3) + 2(-1) = 9$$
$$x + 9 - 2 = 9$$
$$x = 2$$

Solution: $X = 2$, $Y = -3$, $Z = -1$

You can check your solution by plugging the coordinates (2, -3, -1) into the original equations:

$$x - 3y + 2z = 9$$
$$2 - 3(-3) + 2(-1) = 9$$
$$2 + 9 - 2 = 9$$
$$9 = 9$$

$$2x + 5y - z = -10$$
$$2(2) + 5(-3) - (-1) = -10$$
$$4 + (-15) + 1 = -10$$
$$-10 = -10$$

$$-3x + y - 4z = -5$$
$$-3(2) + (-3) - 4(-1) = -5$$
$$-6 + (-3) + 4 = -5$$
$$-5 = -5$$

The *Gauss-Jordan* method involves a little more work with matrices, but the results should be the same. Use row operations to convert the initial augmented matrix into a matrix with ones on the diagonal and zeros elsewhere (except in the solutions column), as shown below.

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 9 \\ 2 & 5 & -1 & -10 \\ -3 & 1 & -4 & -5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right] \text{ which means } \begin{array}{l} x + 0y + 0z = 2 \\ 0x + y + 0z = -3 \\ 0x + 0y + z = -1 \end{array}$$